Volume of random real algebraic submanifolds

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Workshop on random polynomials
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Outline

1. Random real algebraic submanifolds
2. Expectation and variance of the volume
3. About the proofs

Random real algebraic submanifolds

Kostlan–Shub–Smale polynomials

We consider a random $P \in \mathbb{R}_d^{\text{hom}}[X_0, \ldots, X_n]$ distributed according to Kostlan's distribution:

$$P = \sum_{|\alpha|=d} a_\alpha \sqrt{\binom{d}{\alpha}} X^\alpha,$$

where $(a_\alpha)_{|\alpha|=d}$ are independent identically distributed $\mathcal{N}(0,1)$.

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Remark

The family $\left(\sqrt{\binom{d}{\alpha}} X^\alpha\right)_{|\alpha|=d}$ is an orthonormal basis of $\mathbb{R}_d^{\text{hom}}[X_0, \dots, X_n]$ for the inner product:

$$\langle P, Q \rangle = \frac{1}{\pi^{n+1} d!} \int_{\mathbb{C}^{n+1}} P(z) \overline{Q(z)} e^{-\|z\|^2} \, \mathrm{d}z.$$

Algebraic submanifolds of $\mathbb{S}^n$

Let us fix d, n and $r \in \{1, \dots, n\}$.

Definition

Let $P_1, \dots, P_r \in \mathbb{R}_d^{\mathrm{hom}}[X_0, \dots, X_n]$ be independent Kostlan–Shub–Smale polynomials, we set:

$$Z_d = \left(\bigcap_{i=1}^{r} {P_i}^{-1}(0) \right) \cap \mathbb{S}^n.$$

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$$Z_d = \left(\bigcap_{i=1}^{r} P_i{}^{-1}(0) \right) \cap \mathbb{S}^n.$$

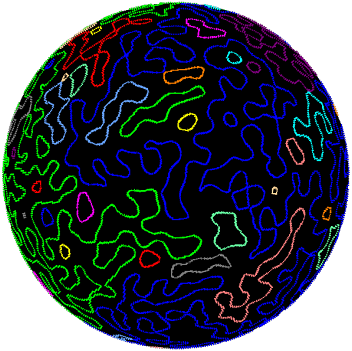
Lemma

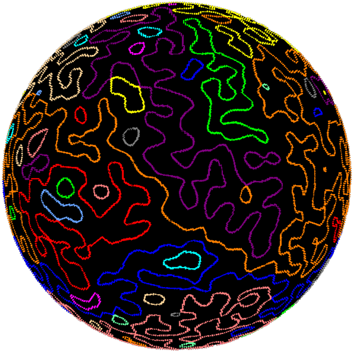
Z_d is almost surely a codimension r smooth submanifold of $\mathbb{S}^n$.

Theorem (Kostlan, 1993)

For all n, r and d, we have: $\mathbb{E}[\mathrm{Vol}(Z_d)] = d^{\frac{r}{2}}\,\mathrm{Vol}(\mathbb{S}^{n-r})$.

Random algebraic curves on $\mathbb{S}^2$





Random algebraic curves of degree 56 on the sphere,
Kostlan–Shub–Smale model.

Pictures by Maria Nastasescu (Brown University).

General setting

$\mathcal{X}$ complex projective manifold of dimension n,

$\mathcal{E} \to \mathcal{X}$ rank r Hermitian vector bundle,

$\mathcal{L} \to \mathcal{X}$ positive Hermitian line bundle.

Assume $\mathcal{X}$, $\mathcal{L}$ and $\mathcal{E}$ are equipped with real structures: $c_{\mathcal{X}}$, $c_{\mathcal{L}}$ and $c_{\mathcal{E}}$.

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Let $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ denote the space of global holomorphic sections of $\mathcal{E} \otimes \mathcal{L}^d \to \mathcal{X}$ invariant with respect to the real structures, that is:

$$(c_{\mathcal{E}} \otimes c_{\mathcal{L}}^d) \circ s \circ c_{\mathcal{X}} = s.$$

Let $M = \mathrm{Fix}(c_{\mathcal{X}})$ denote the real locus of $\mathcal{X}$. We assume $M \neq \emptyset$.
M is a smooth compact manifold without boundary of dimension n.

Zeros of random sections

$\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is finite dimensional and equipped with a natural L^2 inner product.

Definition

Let $s_d \sim \mathcal{N}(0, \mathrm{Id})$ be a random section in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, we set $Z_d = s_d^{-1}(0) \cap M$.

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Example

If $\mathcal{X} = \mathbb{CP}^n$, $\mathcal{L} = \mathcal{O}(1)$ and $\mathcal{E}$ is trivial, we get $M = \mathbb{RP}^n$ and

$$\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) = \left(\mathbb{R}_d^{\mathrm{hom}}[X_0, \dots, X_n]\right)^r.$$

Thus s_d corresponds to r independent Kostlan–Shub–Smale polynomials.

Random Radon measures

Lemma

For d large enough, Z_d is a.s. a codimension r smooth submanifold of M.

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$\mathcal{L} \to \mathcal{X}$ induces a Riemannian metric on $\mathcal{X}$, hence on M and Z_d.

We denote by $|\mathrm{d}V_M|$ the Riemannian volume measure on M and by $|\mathrm{d}V_d|$ the Riemannian volume measure on Z_d.

Z_d defines a continuous linear form on $\left(\mathcal{C}^0(M), \|\cdot\|_\infty\right)$ by:

$$\forall \phi \in \mathcal{C}^0(M), \qquad \langle Z_d \, , \phi \rangle = \int_{Z_d} \phi \, |\mathrm{d}V_d| \, .$$

Expectation and variance of the volume

Expected volume

Let $s_d \sim \mathcal{N}(0, \mathrm{Id})$ in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ and let Z_d denote its real zero set.

Theorem (L., 2014)

For all $\phi \in \mathcal{C}^0(M)$ we have:

$$\mathbb{E}[\langle Z_d\,, \phi \rangle] = d^{\frac{r}{2}} \left(\int_M \phi \, |\mathrm{d}V_M| \right) \frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)} + \|\phi\|_\infty \, O\Big(d^{\frac{r}{2}-1}\Big),$$

where the error term does not depend on ϕ.

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Corollary (Equidistribution of the mean)

We have:

$$d^{-\frac{r}{2}} \mathbb{E}[Z_d] \xrightarrow[d \to +\infty]{} \frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)} \, |\mathrm{d}V_M|$$

as continuous linear forms.

Variance of the volume

Theorem (L., 2016)

If $1 \leqslant r < n$, *then for all* $\phi \in \mathcal{C}^0(M)$ *we have:*

$$\operatorname{Var}(\langle Z_d \,, \phi\rangle) = d^{r-\frac{n}{2}} \left(\int_M \phi^2 \,|\mathrm{d}V_M|\right) \mathcal{I}_{n,r} + o\Big(d^{r-\frac{n}{2}}\Big),$$

where $\mathcal{I}_{n,r}$ *is explicit and* $0 \leqslant \mathcal{I}_{n,r} < +\infty$.

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Corollary (Concentration)

If $1 \leqslant r < n$*, then for all* $\phi \in \mathcal{C}^0(M)$ *we have:*

$$\mathbb{P}\left(\Big|\langle Z_d\,,\phi\rangle - \mathbb{E}[\langle Z_d\,,\phi\rangle]\Big| \geqslant d^{\frac{r}{4}}\right) = O\Big(d^{\frac{r-n}{2}}\Big)\,.$$

The top codimension case

For one Kostlan–Shub–Smale polynomial on $\mathbb{S}^1$ ($n = r = 1$).

Theorem (Kostlan, 1993)

$$\mathbb{E}[\mathrm{card}(Z_d)] = 2\sqrt{d}.$$

Theorem (Dalmao, 2015)

There exists an explicit constant $\sigma^2 > 0$ such that:

$$\mathrm{Var}(\mathrm{card}(Z_d)) \sim \sigma^2\sqrt{d}.$$

Moreover, as $d \to +\infty$,

$$\frac{\mathrm{card}(Z_d) - 2\sqrt{d}}{\sigma d^{\frac{1}{4}}} \Longrightarrow \mathcal{N}(0,1).$$

Equidistribution in probability

Corollary

If $1 \leqslant r < n$*, then for any open subset* $U \subset M$ *we have:*

$$\mathbb{P}\left(Z_d \cap U = \emptyset\right) = O\left(d^{-\frac{n}{2}}\right).$$

Equidistribution in probability

Corollary

If $1 \leqslant r < n$, then for any open subset $U \subset M$ we have:

$$\mathbb{P}\left(Z_d \cap U = \emptyset\right) = O\left(d^{-\frac{n}{2}}\right).$$

Let $\phi_U \in \mathcal{C}^0(M)$ be a function that is positive on U and vanishes on $M \setminus U$.
Let $\varepsilon > 0$ such that for every d large enough, $\mathbb{E}[\langle Z_d \,, \phi_U \rangle] > d^{\frac{r}{2}}\varepsilon$.

$$\begin{aligned}
\mathbb{P}\left(Z_d \cap U = \emptyset\right) &= \mathbb{P}\left(\langle Z_d \,, \phi_U \rangle = 0\right) \\
&\leqslant \mathbb{P}\left(\left|\langle Z_d \,, \phi_U \rangle - \mathbb{E}[\langle Z_d \,, \phi_U \rangle]\right| > d^{\frac{r}{2}}\varepsilon\right) \\
&\leqslant \frac{1}{d^r \varepsilon^2} \operatorname{Var}(\langle Z_d \,, \phi_U \rangle) \\
&= O\left(d^{-\frac{n}{2}}\right).
\end{aligned}$$

About the proofs

Correlation function

A KSS polynomial $P \in \mathbb{R}_d^{\text{hom}}[X_0, \ldots, X_n]$ defines a centered Gaussian process $(P(x))_{x \in \mathbb{S}^n}$ with correlation function $e_d : (x, y) \mapsto \mathbb{E}[P(x)P(y)]$.

Remark

Taking partial derivatives, we get: $\frac{\partial e_d}{\partial x_i}(x, y) = \mathbb{E}\left[\frac{\partial P}{\partial x_i}(x)P(y)\right]$.

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$$
\begin{aligned}
e_d(x, y) &= \sum_{|\alpha|=d=|\beta|} \mathbb{E}[a_\alpha a_\beta] \sqrt{\binom{d}{\alpha}} \sqrt{\binom{d}{\beta}} x^\alpha y^\beta \\
&= \sum_{|\alpha|=d} \binom{d}{\alpha} x^\alpha y^\alpha = (\langle x \,, y \rangle)^d \\
&= \cos\left(\rho(x, y)\right)^d,
\end{aligned}
$$

where ρ is the geodesic distance on $\mathbb{S}^n$.

The Bergman kernel

In the general setting, e_d is the Bergman kernel of $\mathcal{E} \otimes \mathcal{L}^d$.

Theorem (Dai–Liu–Ma, 2006)

The Bergman kernel e_d has a universal scaling limit:

$$e_d(x, y) \simeq \exp\left(-\frac{d}{2}\|x - y\|^2\right),$$

uniformly for (x, y) such that $\rho(x, y) \leqslant K\frac{\log d}{\sqrt{d}}$.

Theorem (Ma–Marinescu, 2015)

There exists $C > 0$ such that, for any $k \in \mathbb{N}$,

$$\|e_d(x, y)\|_{\mathcal{C}^k} = O\Big(d^{\frac{k}{2}} \exp\left(-C\sqrt{d}\rho(x, y)\right)\Big),$$

as $d \to +\infty$, uniformly on $M \times M$.

Kac–Rice formula

Kac-Rice formula

For any $\phi \in \mathcal{C}^0(\mathbb{S}^n)$, we have:

$$\mathbb{E}\left[\int_{Z_d} \phi \left|\mathrm{d}V_d\right|\right] = \frac{1}{\sqrt{2\pi}} \int_{x \in \mathbb{S}^n} \phi(x) \frac{\mathbb{E}\left[\left\|d_x P\right\| \,\middle|\, P(x) = 0\right]}{\sqrt{e_d(x,x)}} \left|\mathrm{d}V_{\mathbb{S}^n}\right|.$$

In the general setting, $x \mapsto e_d(x,x)$ does not vanish for d large enough and we get the same formula.

We need to estimate $\dfrac{\mathbb{E}\left[\left\|d_x P\right\| \,\middle|\, P(x) = 0\right]}{\sqrt{e_d(x,x)}}$ for a given $x \in M$.

Asymptotic of the expectation

$(P(x), d_x P)$ is a centered Gaussian vector with variance

$$\Lambda = \begin{pmatrix} e_d(x,x) & \partial_{y_1} e_d(x,x) & \cdots & \partial_{y_n} e_d(x,x) \\ \partial_{x_1} e_d(x,x) & \partial_{x_1}\partial_{y_1} e_d(x,x) & \cdots & \partial_{x_1}\partial_{y_n} e_d(x,x) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{x_n} e_d(x,x) & \partial_{x_n}\partial_{y_1} e_d(x,x) & \cdots & \partial_{x_n}\partial_{y_n} e_d(x,x) \end{pmatrix}.$$

The distribution of $d_x P$ given that $P(x) = 0$ is a centered Gaussian and its variance only depends on e_d and its derivatives at (x, x).

We get a universal asymptotic for $\dfrac{\mathbb{E}\left[\|d_x P\| \,\middle|\, P(x) = 0\right]}{\sqrt{e_d(x,x)}}$, using the results of Dai, Liu and Ma.

A formula for the variance

$$\begin{aligned}\mathrm{Var}(\langle Z_d\,,\phi\rangle) &= \mathbb{E}\left[\langle Z_d\,,\phi\rangle^2\right] - \mathbb{E}[\langle Z_d\,,\phi\rangle]^2 \\ &= \mathbb{E}\left[\int_{x,y\in Z_d} \phi(x)\phi(y)\,|\mathrm{d}V_d|^2\right] - \mathbb{E}\left[\int_{x\in Z_d} \phi(x)\,|\mathrm{d}V_d|\right]^2.\end{aligned}$$

By Kac–Rice formulas, we get:

$$\mathrm{Var}(\langle Z_d\,,\phi\rangle) = \int_{x,y\in\mathbb{S}^n} \phi(x)\phi(y)\mathcal{D}_d(x,y)\,|\mathrm{d}V_{\mathbb{S}^n}|^2\,,$$

where $\mathcal{D}_d(x,y)$ only depends on e_d and its derivatives at (x,x), (x,y), (y,x) and (y,y).

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where $\mathcal{D}_d(x,y)$ only depends on e_d and its derivatives at (x,x), (x,y), (y,x) and (y,y).

Main problem

$\mathcal{D}_d$ is singular on the diagonal in $\mathbb{S}^n\times\mathbb{S}^n$.

Behaviour of the density $\mathcal{D}_d$

Far from the diagonal

For a good choice of $K > 0$, we have $\mathcal{D}_d(x, y) = O\left(d^{r-\frac{n}{2}-1}\right)$ uniformly on:

$$\left\{(x, y) \in \mathbb{S}^n \times \mathbb{S}^n \;\middle|\; \rho(x, y) \geqslant K \frac{\log d}{\sqrt{d}}\right\}.$$

Near the diagonal

On $\left\{(x, y) \in \mathbb{S}^n \times \mathbb{S}^n \;\middle|\; \rho(x, y) < K \frac{\log d}{\sqrt{d}}\right\}$, we have the following universal scaling limit:

$$\mathcal{D}_d\left(x, x + \frac{z}{\sqrt{d}}\right) \simeq d^r \mathcal{D}(\|z\|),$$

where $\|z\| < K \log d$.

Asymptotic of the variance

$$\mathrm{Var}(\langle Z_d\,,\phi\rangle) \simeq \int_{x\in\mathbb{S}^n}\int_{y\in B(x,K\frac{\log d}{\sqrt{d}})} \phi(x)\phi(y)\mathcal{D}_d(x,y)\,|\mathrm{d}V_{\mathbb{S}^n}|^2$$

$$\simeq d^{-\frac{n}{2}}\int_{x\in\mathbb{S}^n}\left(\int_{\|z\|<K\log d}\phi(x)\phi\left(x+\frac{z}{\sqrt{d}}\right)\mathcal{D}_d\left(x,x+\frac{z}{\sqrt{d}}\right)\mathrm{d}z\right)|\mathrm{d}V_{\mathbb{S}^n}|$$

$$\simeq d^{r-\frac{n}{2}}\left(\int_{x\in\mathbb{S}^n}\phi(x)^2\,|\mathrm{d}V_{\mathbb{S}^n}|\right)\left(\int_{\mathbb{R}^n}\mathcal{D}(\|z\|)\,\mathrm{d}z\right).$$

Almost sure equidistribution

We consider a random sequence of homogeneous polynomials of increasing degree:

$$(P_d)_{d\in\mathbb{N}^*} \in \prod_{d\in\mathbb{N}^*} \mathbb{R}_d^{\text{hom}}[X_0, \ldots, X_n],$$

distributed according to $\mathrm{d}\nu$, the product of the Kostlan distributions.

Corollary

If $n \geqslant 3$, then $\mathrm{d}\nu$-almost surely we have:

$$\forall \phi \in \mathcal{C}^0(\mathbb{S}^n), \qquad \frac{1}{\sqrt{d}} \langle Z_{P_d}, \phi \rangle \xrightarrow[d\to+\infty]{} \frac{\mathrm{Vol}\left(\mathbb{S}^{n-1}\right)}{\mathrm{Vol}\left(\mathbb{S}^n\right)} \int_{\mathbb{S}^n} \phi \left|\mathrm{d}V_{\mathbb{S}^n}\right|.$$

Almost sure equidistribution

Let $\phi \in \mathcal{C}^0(\mathbb{S}^n)$, we have:

$$\mathbb{E}\left[\sum_{d\geqslant 1}\left(\frac{1}{\sqrt{d}}\left(\langle Z_{P_d}\,,\phi\rangle - \mathbb{E}[\langle Z_d\,,\phi\rangle]\right)\right)^2\right] = \sum_{d\geqslant 1}\frac{1}{d}\operatorname{Var}(\langle Z_d\,,\phi\rangle) < +\infty,$$

since $\operatorname{Var}(\langle Z_d\,,\phi\rangle) = O\left(d^{1-\frac{3}{2}}\right)$. Thus $\mathrm{d}\nu$-a.s.

$$\sum_{d\geqslant 1}\left(\frac{1}{\sqrt{d}}\left(\langle Z_{P_d}\,,\phi\rangle - \mathbb{E}[\langle Z_d\,,\phi\rangle]\right)\right)^2 < +\infty,$$

and

$$\frac{1}{\sqrt{d}}\langle Z_{P_d}\,,\phi\rangle \xrightarrow[d\to+\infty]{\mathrm{d}\nu-\text{a.s.}} \frac{\operatorname{Vol}\left(\mathbb{S}^{n-1}\right)}{\operatorname{Vol}\left(\mathbb{S}^{n}\right)}\int_{\mathbb{S}^n}\phi\,|\mathrm{d}V_{\mathbb{S}^n}|\,.$$

We conclude by a separability argument.

The end

Thank you for your attention.